Level dynamics approach to the large deviation statistical characteristic function

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We propose a level dynamics approach to the large deviation statistical characteristic function $\phi(q)$ for temporal series of dynamical variable V, which is the largest eigenvalue of the generalized evolution operator $H_q(\equiv H+qV)$. This is done first by deriving "equations of motion" for the eigenvalues and the eigenstates of H_q with the initial conditions determined by those of H, the true evolution operator for the dynamical variable under consideration, and then by solving these equations. Furthermore, utilizing simple solvable models, it is shown that the eigenvalues and eigenstates satisfy the equations of motion derived in this paper.

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I. INTRODUCTION

It is nowadays well known that the large deviation statistics is the extension of the central limit theorem in the sense that the latter holds in the region of small fluctuation around the highest probability, the large deviation statistics holds in the whole region including large deviations from the long time average. The large deviation statistics has a close connection with the equilibrium statistical mechanics, the multifractal theory of turbulence and chaos, and the probability theory [1–10].

The large deviation statistics is described by the large deviation statistical characteristic function $\phi(q)$, where q is an arbitrary real number and ϕ is a concave function of q. For the time series $\{V_t\}$ of a dynamical variable V, whose time evolution is determined by the operator H, the function $\phi(q)$ is determined by the generalized time evolution operator H_q defined by [3]

$$H_q = H + qV \tag{1}$$

for the continuous time systems such as Markovian stochastic process, Langevin dynamics, and chaotic dynamics in dissipative systems (Appendix A). For discrete time systems, the generalized time evolution operator is given by [3,11–13]

$$H_q = He^{qV} \tag{2}$$

(Appendix B).

Previously, we proposed a method to derive the eigenvalues of the operator H_q by utilizing a finite state approximation for H_q for discrete time dynamics and showed its usefulness [13]. In this method, the function $\phi(q)$ is determined by one of the poles of a certain resolvent, i.e., the closest pole to the origin. The fundamental aim of the present paper is to formulate the method of determination of $\phi(q)$ in a different way. Is it possible to derive $\phi(q)$ with the data on the eigenstates of the conventional evolution operator H? The present paper is concerned with this question. In other words, "Is it possible to solve the eigenvalue problem of H_q when the eigenvalue problem of H is solved?." The answer is yes. The present problem is quite similar to the so called level dynamics in quantum mechanics.

Two decades ago, it was found that the eigenvalue problem of the Hamiltonian $H=H_0+\tau V$ for a quantum bound system which is classically nonintegrable, where H_0 and V are a classically integrable Hamiltonian and a nonintegrable part, respectively, and τ represents for the strength of nonintegrability, is formulated as coupled "dynamical" equations for energy levels and eigenfunctions by regarding τ as a time [14–17]. This is known as the level dynamics of eigenvalue problem. The problem under consideration in this paper is identical to solving the nonintegrable Hamiltonian except for the fact that in the present case the variable is not Hermitian in contrast to that Hamiltonian operators in quantum mechanics are Hermitian. The fundamental aim of the present paper is to propose a method to obtain the large deviation statistical characteristic function $\phi(q)$ by solving the eigenvalue problem of H_q when the eigenvalue problem of H is solved by applying the idea developed in the eigenvalue problem in quantum mechanics [14–17].

The present paper is constructed as follows. In Sec. II, we derive a set of "equations of motion" for eigenvalues and eigenstates of the generalized time evolution operator H_q. It will be shown in Sec. III that simple examples rigorously solved satisfy the proposed equations of motion. Concluding remarks are given in Sec. IV. In Appendix A, we give a brief review of the large deviation statistics and show the relation between the large deviation statistical characteristic function and the largest eigenvalue of the generalized time evolution operator H_q. In Appendix B, the present method is extended to discrete-time systems such as discrete-time Markovian stochastic processes and chaotic map systems.

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II. DETERMINATION OF LARGE DEVIATION STATISTICAL CHARACTERISTIC FUNCTION

A. "Equations of motion" for eigenvalues and eigenfunctions of the generalized evolution operator H_q

Let H be a time evolution operator such as the transition matrix and the Fokker-Planck operator [18] and V is the physical variable. Since we are considering dissipative dynamical systems including stochastic process, the time evolution operator H, whose components are real, is not a Hermitian operator. The operator V has real components. As shown in Appendix A, the large deviation statistical characteristic function for continuous time systems is determined by the largest eigenvalue of the generalized evolution operator H_q given in Eq. (1). For discrete-time systems, see Appendix B.

The right and left eigenvalue problems of the operator H_q are, respectively, written as

$$H_q|n\rangle = \lambda_n(q)|n\rangle, \tag{3}$$

$$\langle m|H_q = \underline{\lambda}_m(q)\langle m|, \tag{4}$$

$(n=1,2,3,\ldots)$. λ_n and $\underline{\lambda}_m$ are eigenvalues, and a ket $|n\rangle$ and a bra $\langle m|$ are eigenstates. Throughout the paper, we assume that all the eigenstates are discrete and are nondegenerate. One should note that eigenvalues $\{\lambda_n\}$ can be complex numbers since H is not Hermitian. Multiplying $\langle m|$ and $|n\rangle$, respectively, to Eqs. (3) and (4), we obtain

$$\langle m|H_q|n\rangle = \lambda_n\langle m|n\rangle, \tag{5}$$

$$\langle m|H_q|n\rangle = \underline{\lambda}_m\langle m|n\rangle. \tag{6}$$

From these equations, the following relation follows:

$$(\lambda_n - \underline{\lambda}_m)\langle m|n\rangle = 0. \tag{7}$$

This equation implies that there exists an m satisfying $\underline{\lambda}_m =\lambda_n$. Without loss of generality, this m is numbered as n. Furthermore, with the normalization of eigenstates, we get

$$\underline{\lambda}_n = \lambda_n, \tag{8}$$

$$\langle m|n\rangle = \delta_{m,n}, \tag{9}$$

where $\delta_{m,n}$ is the Kronecker delta. It should be noted that Eq. (9) is invariant under the transformation $\{|n\rangle \to B_n|n\rangle\}$ and $\{\langle m| \to B_m^{-1}\langle m|\}$ with a set of arbitrary constants $\{B_n\}$. Without loss of generality, we put $\max_n \mathrm{Re}\,\lambda_n(q) = \mathrm{Re}\,\lambda_1(q)$. As will be shown later, $\lambda_1(q)$ is real. Other eigenvalues except λ_1 can be generally complex because of the non-Hermitian property of the operator H_q.

Differentiating Eq. (5) with respect to q, and noting the relation

$$\left(\frac{d}{dq}\langle m|\right)|n\rangle + \langle m|\frac{d}{dq}|n\rangle = 0, \tag{10}$$

we obtain

$$(\lambda_m - \lambda_n)\langle m|\frac{d}{dq}|n\rangle + V_{mn} = \frac{d\lambda_n}{dq}\delta_{m,n}, \tag{11}$$

where we introduced the notation

$$V_{mn} \equiv \langle m|V|n\rangle. \tag{12}$$

Equation (11) is rewritten as

$$\frac{d\lambda_n(q)}{dq} = V_{nn}, \tag{13}$$

$$(\lambda_m - \lambda_n)\langle m|\frac{d}{dq}|n\rangle = V_{nn}\delta_{m,n} - V_{mn}. \tag{14}$$

By assuming that eigenstates $\{|n\rangle\}$ constitute a complete set of bases, the derivative $d|n\rangle/dq$ can be expanded as a linear combination of ket vectors. If $d|n\rangle/dq$ contains the $|n\rangle$ component, then, by appropriately choosing a set of bases $\{B_n(q)|n\rangle\}$ with scalar quantities $\{B_n\}$, it is always possible to choose $\{|k\rangle\}$ in such a way that $d|n\rangle/dq$ does not contain the component $|n\rangle$. Inserting the expansion $d|n\rangle/dq =\sum_{k(\neq n)}c_{nk}|k\rangle$ with the expansion coefficients c_{nk} into Eq. (11), one obtains $c_{nk}=V_{kn}/(\lambda_n-\lambda_k)$ and therefore

$$\frac{d}{dq}|n\rangle = \sum_{k(\neq n)}\frac{V_{kn}}{\lambda_n - \lambda_k}|k\rangle. \tag{15}$$

Furthermore, it is easy to observe that the following expansion holds:

$$\frac{d}{dq}\langle m| = \sum_{k(\neq m)}\frac{V_{mk}}{\lambda_m - \lambda_k}\langle k|. \tag{16}$$

It should be noted that if $\{|n\rangle\}$ and $\{\langle m|\}$ constitute a complete set of bases, then an arbitrary state $|\psi\rangle$ is expanded as $|\psi\rangle=\sum_n\langle n|\psi\rangle|n\rangle=\sum_n|n\rangle\langle n|\psi\rangle$. Thus, we obtain

$$\sum_n |n\rangle\langle n| = 1. \tag{17}$$

This is called the completeness condition of eigenstates.

Differentiating Eq. (12), we obtain

$$\frac{dV_{mn}(q)}{dq} = \sum_{k(\neq m)}\frac{V_{mk}V_{kn}}{\lambda_m - \lambda_k} + \sum_{k(\neq n)}\frac{V_{mk}V_{kn}}{\lambda_n - \lambda_k}. \tag{18}$$

It should be noted that if the parameter q is regarded as a fictitious time, then Eqs. (13), (15), (16), and (18) are identical to "equations of motion" for the eigenvalues $\{\lambda_n(q)\}$, eigenstates $\{|n\rangle(q),\langle m|(q)\}$, and the elements $\{V_{mn}(q)\}$ under the initial conditions $\{\lambda_n(0)\}$, $\{|n\rangle(0),\langle m|(0)\}$, and $\{V_{mn}(0)\}$ determined by the operator H. Therefore, numerically solving these equations of motion with the initial condition determined by the eigenstates of H, we can solve the eigenvalue problem of H_q for any q.

The large deviation statistical characteristic function $\phi(q)$ is the largest eigenvalues, i.e., $\phi(q)=\max_n\{\mathrm{Re}\,\lambda_n(q)\}$. Without loss of generality, we put $\phi(q)=\mathrm{Re}\,\lambda_1(q)=\lambda_1(q)$. If $\lambda_1(q)$ is complex, the characteristic function $Z_q(t)$ can be negative. In order that $Z_q(t)$ is positive for any t, the largest eigenvalue $\lambda_1(q)$ should be real. Therefore, noting that the largest eigenvalue of H vanishes, we get the initial condition $\lambda_1(0)=0$. The equation of motion $d\lambda_1(q)/dq=V_{11}(q)$ turns out to be identical to the relation $\phi'(q)=v(q)$ given in Appendix A. Furthermore, the fluctuation spectrum obtained by Eq. (A4) is written as

$$S[V_{11}(q)] = qV_{11}(q) - \lambda_1(q). \tag{19}$$

This relation implies that the fluctuation spectrum $S(V_{11})$ is numerically obtained by plotting $(qV_{11}-\lambda_1)$ as a function of V_{11} for different values of q.

For a moment, let us consider the case in which the number of eigenstates is finite, which is denoted as N. In this case, the quantity

$$h \equiv \sum_{n}^{N} V_{nn}(q) \tag{20}$$

turns out to be a conserved quantity, i.e., is independent of q. This fact can be immediately proved with Eq. (18). As a result, the following equality holds:

$$\sum_{n}^{N} [\lambda_n(q) - \lambda_n(0)] = hq. \tag{21}$$

Let us consider the space spanned by $\{\lambda_n\}$ and $\{V_{mn}\}$. The rate $R(q)$ of contraction or divergence of the volume of a tiny region in this space is given by

$$\begin{aligned} R(q) &\equiv \sum_{n} \frac{\partial}{\partial \lambda_n}\left(\frac{d\lambda_n}{dq}\right) + \sum_{m,n} \frac{\partial}{\partial V_{mn}}\left(\frac{dV_{mn}}{dq}\right) \\ &= -\sum_{m\neq n} \frac{V_{mm}-V_{nn}}{\lambda_m-\lambda_n} \\ &= -\frac{d}{dq}\sum_{m\neq n} \ln|\lambda_m-\lambda_n|. \end{aligned} \tag{22}$$

In the following subsections, we reformulate the above approach to concrete dynamics.

B. Finite-state Markovian process

Let us consider the finite-state Markovian process described by the evolution equation

$$\frac{\partial \boldsymbol{P}(t)}{\partial t} = \hat{H}\boldsymbol{P}(t), \tag{23}$$

where $\boldsymbol{P}(t)=[P_1(t),P_2(t),\ldots,P_N(t)]^T$ with $P_j(t)$ being the probability that the state is in the jth state at time t. $\hat{H}$ is the transition matrix with the jth element H_{jk} $(\sum_{j=1}^{N}H_{jk}=1)$. We consider the time series of the variable V_t which takes the value a_j if the system is in the jth state. The large deviation statistical quantity for the time series $\{V_t\}$ is determined by the largest eigenvalue of the matrix [3]

$$\hat{H}_q = \hat{H} + q\hat{V} \tag{24}$$

(Appendix A). Here $\hat{V}$ is the matrix with the jk element $V_{jk} = a_j\delta_{j,k}$.

Let $\boldsymbol{e}_n$ and $\underline{\boldsymbol{e}}_m$ be, respectively, the right and left eigenvalues of $\hat{H}_q$. Eigenvalue equations are written as

$$(\hat{H}_q\boldsymbol{e}_n)_j \equiv \sum_{k=1}^{N} (\hat{H}_q)_{jk}(\boldsymbol{e}_n)_k = \lambda_n(q)(\boldsymbol{e}_n)_j, \tag{25}$$

$$(\underline{\boldsymbol{e}}_m\hat{H}_q)_j \equiv \sum_{k=1}^{N} (\underline{\boldsymbol{e}}_m)_k(\hat{H}_q)_{kj} \equiv (\underline{\hat{H}}_q\underline{\boldsymbol{e}}_m)_j = \lambda_m(q)(\underline{\boldsymbol{e}}_m)_j. \tag{26}$$

Here we have defined the adjoint operator $\underline{\hat{H}}_q$ by $\underline{\hat{H}}_q=\hat{H}_q^T$, T representing the transpose. The orthogonality of eigenvectors are written as

$$\underline{\boldsymbol{e}}_m \cdot \boldsymbol{e}_n = \sum_{j=1}^{N} (\underline{\boldsymbol{e}}_m)_j(\boldsymbol{e}_n)_j = \delta_{m,n}. \tag{27}$$

The equality

$$\underline{\boldsymbol{e}}_m\hat{H}_q\boldsymbol{e}_n = \lambda_n\delta_{m,n} \tag{28}$$

yields the evolution equations (13) for the eigenvalues $\{\lambda_n\}$ and (18) for the matrix elements $\{V_{mn}\}$ with

$$V_{mn} = \underline{\boldsymbol{e}}_m\hat{V}\boldsymbol{e}_n = \sum_{j=1}^{N} a_j(\underline{\boldsymbol{e}}_m)_j(\boldsymbol{e}_n)_j. \tag{29}$$

The completeness condition corresponding to Eq. (17) is written as

$$\sum_{n=1}^{N} (\underline{\boldsymbol{e}}_n)_k(\boldsymbol{e}_n)_j = \delta_{j,k}. \tag{30}$$

C. Langevin dynamics

Let us consider the Langevin dynamics

$$\dot{\boldsymbol{A}}(t) = \boldsymbol{F}[\boldsymbol{A}(t),\boldsymbol{R}(t)], \tag{31}$$

where $\boldsymbol{A}(t)$ is the Langevin variable and $\boldsymbol{R}(t)$ is the white Langevin random force. The corresponding master equation is written as the Markovian form [18]

$$\frac{\partial P(\boldsymbol{a},t)}{\partial t} = H(\boldsymbol{a})P(\boldsymbol{a},t) \tag{32}$$

with the master operator $H(\boldsymbol{a})$. If we consider the time evolution of the scalar variable $V[\boldsymbol{A}(t)]\equiv V_t$, the large deviation statistics of the finite-time average $T^{-1}\int_0^T V_s ds$ is determined by the largest eigenvalue of the generalized evolution operator [19]

$$H_q(\boldsymbol{a}) = H(\boldsymbol{a}) + qV(\boldsymbol{a}) \tag{33}$$

(Appendix A).

Let $f_n(\boldsymbol{a})$ and $\underline{f}_m(\boldsymbol{a})$ are, respectively, the right and left eigenfunctions of H_q. The eigenvalue equations are written as

$$(H_qf_n)(\boldsymbol{a}) \equiv \int d\boldsymbol{b}(H_q)_{\boldsymbol{a},\boldsymbol{b}}f_n(\boldsymbol{b}) = \lambda_nf_n(\boldsymbol{a}), \tag{34}$$

$$(\underline{f}_m H_q)(\boldsymbol{a}) \equiv \int d\boldsymbol{b}\, \underline{f}_m(\boldsymbol{b})(H_q)_{\boldsymbol{b},\boldsymbol{a}} = (\underline{H}_q \underline{f}_m)(\boldsymbol{a})$$

$$= \lambda_m \underline{f}_m(\boldsymbol{a}), \qquad (35)$$

where the kernels $(H_q)_{\boldsymbol{a},\boldsymbol{b}}$ and $(\underline{H}_q)_{\boldsymbol{a},\boldsymbol{b}}$ have been defined by

$$(H_q)_{\boldsymbol{a},\boldsymbol{b}} = H_q(\boldsymbol{a})\delta(\boldsymbol{a}-\boldsymbol{b}), \qquad (36)$$

$$(\underline{H}_q)_{\boldsymbol{a},\boldsymbol{b}} = \underline{H}_q(\boldsymbol{a})\delta(\boldsymbol{a}-\boldsymbol{b}) = (H_q)_{\boldsymbol{b},\boldsymbol{a}}, \qquad (37)$$

where the adjoint operator $\underline{H}_q$ of H_q satisfies $\int d\boldsymbol{a} G_1(\boldsymbol{a})H_q(\boldsymbol{a})G_2(\boldsymbol{a}) = \int d\boldsymbol{a}[\underline{H}_q(\boldsymbol{a})G_1(\boldsymbol{a})]G_2(\boldsymbol{a})$. The orthogonality of eigenfunctions is written as

$$\int d\boldsymbol{a}\, \underline{f}_m(\boldsymbol{a}) f_n(\boldsymbol{a}) = \delta_{m,n}. \qquad (38)$$

The equality

$$\int d\boldsymbol{a}\, \underline{f}_m(\boldsymbol{a}) H_q(\boldsymbol{a}) f_n(\boldsymbol{a}) = \lambda_n(q)\delta_{m,n} \qquad (39)$$

yields the evolution equations (13) for the eigenvalues $\{\lambda_n\}$ and (18) for the elements $\{V_{mn}\}$ defined by

$$V_{mn}(q) \equiv \int d\boldsymbol{a}\, \underline{f}_m(\boldsymbol{a}) V(\boldsymbol{a}) f_n(\boldsymbol{a}). \qquad (40)$$

The completeness condition (17) is written as

$$\sum_n \underline{f}_n(\boldsymbol{b}) f_n(\boldsymbol{a}) = \delta(\boldsymbol{a}-\boldsymbol{b}). \qquad (41)$$

III. APPLICATION TO SIMPLE STOCHASTIC PROCESSES

A. Two-state Markovian process

Let us consider the two-state Markovian process with the transition matrix

$$\hat{H} = \begin{pmatrix} -\kappa_1 & \kappa_2 \\ \kappa_1 & -\kappa_2 \end{pmatrix} \qquad (42)$$

($\kappa_1 > 0, \kappa_2 > 0$). The generalized transition matrix $\hat{H}_q$ and its adjoint matrix are given by

$$\hat{H}_q = \begin{pmatrix} -\kappa_1 + a_1 q & \kappa_2 \\ \kappa_1 & -\kappa_2 + a_2 q \end{pmatrix} = \underline{\hat{H}}_q^T. \qquad (43)$$

The eigenvalues $\{\lambda_n\}$ and the elements $\{V_{mn}\}$ are obtained by solving Eqs. (13) and (18) with the initial conditions for them determined by H. On the other hand, since the eigenvalue problem of the matrix (43) is easily solved, we prove that those solutions satisfy Eqs. (13) and (18) instead of directly solving the equations of motion.

It is easy to solve the eigenvalue problem of H_q. The resulting eigenvalues $\{\lambda_n(q)\}$ and eigenvectors $\{\boldsymbol{e}_n(q), \underline{\boldsymbol{e}}_n(q)\}$ of $\hat{H}_q$ are [3]

$$\lambda_{1,2}(q) = \frac{1}{2}[-(\kappa_1+\kappa_2) + (a_1+a_2)q \pm \sqrt{Q^2+4\kappa_1\kappa_2}], \qquad (44)$$

$$\boldsymbol{e}_{1,2}(q) = A_{1,2}\begin{pmatrix} \kappa_2 \\ \dfrac{Q \pm \sqrt{Q^2+4\kappa_1\kappa_2}}{2} \end{pmatrix}, \qquad (45)$$

$$\underline{\boldsymbol{e}}_{1,2}(q) = A_{1,2}\left(\kappa_1, \frac{Q \pm \sqrt{Q^2+4\kappa_1\kappa_2}}{2}\right), \qquad (46)$$

where

$$Q = \kappa_1 - \kappa_2 - (a_1 - a_2)q, \qquad (47)$$

$$A_{1,2} = \sqrt{\frac{\sqrt{Q^2+4\kappa_1\kappa_2} \mp Q}{2\kappa_1\kappa_2\sqrt{Q^2+4\kappa_1\kappa_2}}}. \qquad (48)$$

Furthermore, the elements $\{V_{mn}\}$ are obtained as

$$V_{11}(q) = \frac{1}{2}\left[a_1 + a_2 - \frac{(a_1-a_2)Q}{\sqrt{Q^2+4\kappa_1\kappa_2}}\right], \qquad (49)$$

$$V_{12}(q) = V_{21}(q) = \frac{(a_1-a_2)\sqrt{\kappa_1\kappa_2}}{\sqrt{Q^2+4\kappa_1\kappa_2}}, \qquad (50)$$

$$V_{22}(q) = \frac{1}{2}\left[a_1 + a_2 + \frac{(a_1-a_2)Q}{\sqrt{Q^2+4\kappa_1\kappa_2}}\right]. \qquad (51)$$

It is easy to confirm that Eqs. (44)–(46) and (49)–(51) satisfy the equations of motion (13), (15), (16), and (18). One finds that $V_{11}(q)+V_{22}(q) = a_1+a_2$ and $\lambda_1(q)+\lambda_2(q) = -(\kappa_1+\kappa_2) + (a_1+a_2)q$. The rate of contraction of a volume in the state space is given by

$$R(q) = -\frac{2(a_1-a_2)Q}{Q^2+4\kappa_1\kappa_2} = \frac{d}{dq}\ln(Q^2+4\kappa_1\kappa_2). \qquad (52)$$

Therefore, we obtain $R(q) > 0 (<0)$ for $q < q_* (> q_*)$, where $q_* \equiv (\kappa_1-\kappa_2)/(a_1-a_2)$.

The large deviation statistical characteristic functions are obtained as

$$\phi(q) = \lambda_1(q) = \frac{1}{2}[-(\kappa_1+\kappa_2) + (a_1+a_2)q + \sqrt{Q^2+4\kappa_1\kappa_2}], \qquad (53)$$

$$S(v) = \frac{\kappa_1+\kappa_2}{2} + \frac{\kappa_1-\kappa_2}{a_1-a_2}\left(v - \frac{a_1+a_2}{2}\right) - \frac{2\sqrt{\kappa_1\kappa_2}}{|a_1-a_2|}\sqrt{(a_1-v)(v-a_2)}. \qquad (54)$$

It should be noted that the fluctuation spectrum has a parabolic form $S(v) = (\kappa_1+\kappa_2)(v-v_0)^2/4\sqrt{\kappa_1\kappa_2}$ near $v = v_0 \equiv \phi'(0) = (a_1\kappa_1 + a_2\kappa_2)/(\kappa_1+\kappa_2)$, the long time average of V_t. The deviation of $S(v)$ from the parabolic form near its minimum describes the large deviation statistics.

B. Ornstein-Uhlenbeck process

The time evolution operator $H(a)$ and its adjoint operator $\underline{H}(a)$ for the Ornstein-Uhlenbeck process are given by

$$H(a)G(a) = \frac{\partial}{\partial a}\left[\left(\gamma a + D\frac{\partial}{\partial a}\right)G(a)\right], \tag{55}$$

$$\underline{H}(a)G(a) = \left(-\gamma a + D\frac{\partial}{\partial a}\right)\frac{\partial G(a)}{\partial a}. \tag{56}$$

The corresponding generalized evolution operator $H_q(a)$ is given by

$$H_q(a) = H(a) + qV(a). \tag{57}$$

Its adjoint operator is given by $\underline{H}_q(a)=\underline{H}(a)+qV(a)$. In the following, we consider the two cases, $V(a)=a$ and a^2. In these cases, the eigenvalue problems of H_q are rigorously solved [20].

Case A: $V(a)=a$. By solving the eigenvalue problem of H_q, the eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{f_n(a),\underline{f}_n(a)\}$ are solved to yield [20]

$$\lambda_n(q) = -(n-1)\gamma + \frac{Dq^2}{\gamma^2}, \tag{58}$$

$$f_n(a) = B_n\sqrt{\frac{\sqrt{\gamma/2\pi D}}{2^{n-1}(n-1)!}}e^{-(\gamma/2D)a^2+(q/\gamma)a} \times \mathrm{He}_{n-1}\left[\sqrt{\frac{\gamma}{2D}}\left(a-\frac{2Dq}{\gamma^2}\right)\right], \tag{59}$$

$$\underline{f}_m(a) = B_m^{-1}\sqrt{\frac{\sqrt{\gamma/2\pi D}}{2^{m-1}(m-1)!}}e^{(q/\gamma)a-2Dq^2/\gamma^3} \times \mathrm{He}_{m-1}\left[\sqrt{\frac{\gamma}{2D}}\left(a-\frac{2Dq}{\gamma^2}\right)\right] \tag{60}$$

$(n=1,2,3,\ldots)$, where $\{\mathrm{He}_n(x)\}$ are the Hermite polynomials. The coefficient B_n is chosen in such a way that $df_n(q)/dq$ and $d\underline{f}_m(q)/dq$ do not contain the components $f_n(q)$ and $\underline{f}_m(q)$, respectively. Furthermore, we get

$$V_{mn}(q) = \frac{2Dq}{\gamma^2}\delta_{m,n} + \sqrt{\frac{2D}{\gamma}\frac{2^{m-1}(m-1)!}{2^{n-1}(n-1)!}} \times \left[\frac{1}{2}\delta_{m-1,n} + (n-1)\delta_{m-1,n-2}\right]. \tag{61}$$

One can confirm that Eqs. (58) and (61) satisfy the equations of motion (13) and (18).

The large deviation statistical characteristic functions are thus obtained as

$$\phi(q) = \lambda_1(q) = \frac{Dq^2}{\gamma^2}, \quad S(v) = \frac{\gamma^2}{4D}v^2. \tag{62}$$

Since the Langevin variable $A(t)$ is Gaussian, the coarse-grained variable $\bar{V}_t$ is Gaussian. Due to this fact, the fluctuation spectrum $S(v)$ has a parabolic form in the whole region of v.

Case B: $V(a)=a^2$. By solving the eigenvalue problem of H_q, the eigenvalues and eigenfunctions are obtained as [20]

$$\lambda_n(q) = \gamma\left[-(n-1)\sqrt{1-\frac{q}{q_c}} + \frac{1}{2}\left(1-\sqrt{1-\frac{q}{q_c}}\right)\right], \tag{63}$$

$$f_n(q) = N_n e^{-(\gamma/4D)(\sqrt{1-q/q_c}+1)a^2}\mathrm{He}_{n-1}(\beta a), \tag{64}$$

$$\underline{f}_m(a) = N_m e^{-(\gamma/4D)(\sqrt{1-q/q_c}-1)a^2}\mathrm{He}_{m-1}(\beta a) \tag{65}$$

$(n=1,2,3,\ldots)$, where

$$q_c = \frac{\gamma^2}{4D}, \quad \beta = \sqrt{\frac{\gamma}{2D}\sqrt{1-\frac{q}{q_c}}},$$

$$N_n = \sqrt{\frac{\beta}{\sqrt{\pi}2^{n-1}(n-1)!}}. \tag{66}$$

It should be noted that the eigenvalue problem of the generalized evolution operator H_q is valid for $q<q_c$. The existence of the characteristic value q_c is due to the Gaussian form of the steady state probability density $P_*(a)$. The elements $\{V_{mn}\}$ are obtained as

$$V_{mn}(q) = \frac{2D}{\gamma}\sqrt{\frac{1}{1-q/q_c}\frac{2^{m-1}(m-1)!}{2^{n-1}(n-1)!}}\left[\frac{1}{4}\delta_{m-1,n+1} + \left(n-\frac{1}{2}\right)\delta_{m-1,n-1} + (n-1)(n-2)\delta_{m-1,n-3}\right]. \tag{67}$$

It is confirmed that Eqs. (63) and (67) satisfy the equations of motion (13) and (18).

The large deviation statistical characteristic functions are obtained as [20]

$$\phi(q) = \lambda_1(q) = \frac{\gamma}{2}\sqrt{1-\frac{q}{q_c}}, \tag{68}$$

$$S(v) = \frac{\gamma}{4}\bar{S}\left(\frac{v}{v_0}\right), \tag{69}$$

where $v_0=D/\gamma$ and $\bar{S}(x)$ is the scaling function defined by

$$\bar{S}(x) = \frac{1}{x}(x-1)^2. \tag{70}$$

By reflecting the fact that the observed variable is always positive, the fluctuation spectrum diverges for $v\leq 0$.

IV. CONCLUDING REMARKS

In the present paper, we developed a theory to finding the large deviation statistical characteristic function $\phi(q)$ by solving the eigenvalue problem of the generalized time evolution operator H_q under the assumption that the eigenvalue problem for the time evolution operator H is solved. The formalism is same as the level dynamics in quantum mechanics except the fact the time evolution operator in the present case is not Hermitian. Namely, by deriving the "equations of motion" for the eigenvalues $\{\lambda_n\}$ and the elements $\{V_{mn}\}$ by regarding the parameter q as the fictitious time with the initial conditions of eigenvalues and eigenstates determined by H, the $\phi(q)$ is determined by the largest eigenvalue among $\{\lambda_n(q)\}$. Furthermore, in the present paper, by applying the present method to simple problems which are already solved, it was confirmed that the present equations of motion reproduce the known results. It is expected that the present approach is useful by numerically finding $\phi(q)$ by solving the "equations of motion." In numerically solving the equations of motion, we have to truncate the eigenstates, provided that there exists an infinite number of eigenstates. We have no knowledge about how to reduce the number of eigenstates. In future, it is desired to develop a study on carrying out concrete numerical integration of the equations of motion.

It is quite important to note that the present dynamical equations for the eigenvalues $\{\lambda_n\}$ and the elements V_{mn} have universal structures irrespective of the detail of a system under consideration. The detail of the system is incorporated in the initial condition. It is well known that the system behavior of a dynamical equation in Hamiltonian systems generally depends on the initial condition and can show either periodic or chaotic behavior. In this sense, the present situation is similar to that in Hamiltonian systems.

Twenty years ago, Nakamura and Lakshmanan [17] showed that the level dynamics of the perturbed operator in quantum mechanics reduces to a completely integrable Calogero-Moser system for the "particle positions" $\{x_n(\equiv\lambda_n)\}$ in one dimension. On the other hand, since the eigenvalues of H_q in the present dynamics are generally complex due to its non-Hermitianity, by putting

$$\lambda_n = x_n + iy_n, \tag{71}$$

$x_n(=\mathrm{Re}\,\lambda_n)$ and $y_n(=\mathrm{Im}\,\lambda_n)$ may be regarded, respectively, as the x and y components of the nth particle. Although the present level dynamics may be written in the form of the particle dynamics in two dimensions, it is expected that they are different from the Calogero-Moser system. It is interesting to clarify the mathematical structure of the present dynamics. Further studies of the dynamical equations derived in the present paper including the problem of integrability and nonintegrability of them are needed.

Finally, let us add another possibility of application of the present method to solve eigenvalue problems in a different context. Namely, the present method to solving the eigenvalue problem of the operator H_1 from that of a different, solvable H_0 may be used as follows. Let H_0 and H_1 be linear operators. Define the operator H_q by

$$H_q = H_0 + q(H_1 - H_0). \tag{72}$$

Numerically integrating for $0 \le q \le 1$ the equations of motion for the eigenvalues $\{\lambda_n\}$, the elements $\{V_{mn}[=(H_1-H_0)_{mn}]\}$, and eigenstates $\{|n\rangle, \langle m|\}$ of H_q under their initial conditions determined by H_0, one can get the eigenvalues and the eigenfunctions of H_1. Applications in concrete systems will be reported in future.

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APPENDIX A: BRIEF REVIEW OF LARGE DEVIATION STATISTICS

Let V_s be the value of a statistically steady random time series $\{V_s\}$ at time s, and let T be its correlation time. The finite-time average over a finite time span t,

$$\bar{V}_t = \frac{1}{t}\int_{t_0}^{t_0+t} V_s ds \tag{A1}$$

is a fluctuating variable. For $t \gg T$, the statistical independence of fluctuations implies that the probability density $P_t(v)$ of $\bar{V}_t$ asymptotically takes the form

$$P_t(v) \sim e^{-S(v)t} \tag{A2}$$

for large t. Here $S(v)$ is independent of t and is a concave function of v. The explicit form of $S(v)$ characterizes the statistics of time series. Apparently, the inequality $S(v) \ge 0$ holds. Owing to the stationarity of the time series, the probability density $P_t(v)$ approaches the distribution $\delta(v-\langle V\rangle)$ as $t \to \infty$, where $\langle V\rangle \equiv \bar{V}_\infty$ is the conventional long-time average. The function $S(v)$ thus shows how fluctuations from $V=\langle V\rangle$ decay as the averaging time t is increased.

By applying the asymptotic form of $P_t(v)$, the characteristic function $Z_q(t)$ has the asymptotic form [2,3]

$$Z_q(t) \equiv \langle e^{qt\bar{V}_t}\rangle \sim e^{\phi(q)t} \tag{A3}$$

for large t. Here, $\phi(q)$ is independent of t and is related to $S(v)$ through

$$\phi(q) = -\min_v[S(v) - qv]. \tag{A4}$$

The value of v which minimizes $[\cdots]$ satisfies $v(q)=\phi'(q)$. On the other hand, from the definition of $\phi(q)$ [Eq. (A3)], one obtains

$$v(q) = \lim_{t\to\infty}\left\langle \bar{V}_t \frac{e^{qt\bar{V}_t}}{Z_q(t)}\right\rangle. \tag{A5}$$

The value $v(0)$ is identical to the conventional long-time average $\bar{V}_\infty$.

Generally, $v(q)$ differs from the long time average, since $v(q)$ is the average with the weight factor $e^{qt\bar{V}_t}/Z_q(t)$. Due to this factor, changing the parameter q, one is able to observe a fluctuation $\bar{V}_t - v(0)$ by magnifying its realization probability. In particular, if $\{\bar{V}_t\}$ is Gaussian, then $v(q)$ is simply a linear function of q. The deviation from linear dependence on q near 0 quantifies the non-Gaussian nature of the time series. In this framework of the analysis, one can characterize a coarse-grained quantity $\bar{V}_t$ for large t in terms of the large deviation statistical characteristic functions $S(v)$ and $\phi(q)$. The function $S(v)$ is called the fluctuation spectrum. Hereafter, by using finite-state Markovian process, Langevin dynamics and chaotic dynamics, it is shown that the characteristic function $\phi(q)$ is given by the largest eigenvalue of the operator H_q [Eq. (1)].

1. Finite-state Markovian process

Consider the N-state Markov process whose time evolution is given by

$$\frac{\partial \boldsymbol{P}(t)}{\partial t} = \hat{H}\boldsymbol{P}(t), \tag{A6}$$

where $\boldsymbol{P}(t)=[P_1(t),P_2(t),\ldots,P_N(t)]^T$, $P_j(t)$ being the probability that the system is in the jth state at time t $[\sum_{j=1}^N P_j(t)=1]$. $\hat{H}$ is the transition matrix with the jk element H_{jk} $(\sum_{j=1}^N H_{jk}=1)$. For the observed variable V_t, which takes the value a_j when the system is in the jth state, the characteristic function is given by [3,10]

$$Z_q(t) = \left\langle \exp\left(q\int_{t_0}^{t_0+t} V_s ds \right) \right\rangle = \sum_{j=1}^{N} (e^{t\hat{H}_q}\boldsymbol{P}_*)_j, \tag{A7}$$

where the brackets denote the long time average and therefore it is identical to the average over the steady state probability $\boldsymbol{P}_*$ $(\hat{H}\boldsymbol{P}_*=\boldsymbol{0})$. $\hat{H}_q$ is defined by [3,10]

$$\hat{H}_q = \hat{H} + q\hat{V}. \tag{A8}$$

One thus finds that the large deviation statistical characteristic function $\phi(q)$ is identical to the largest eigenvalue of $\hat{H}_q$.

2. Langevin dynamics

Let us consider the Langevin dynamics

$$\dot{\boldsymbol{A}}(t) = \boldsymbol{F}[\boldsymbol{A}(t),\boldsymbol{R}(t)], \tag{A9}$$

where $\boldsymbol{A}(t)$ is the physical variable and $\boldsymbol{R}(t)$ is the white Langevin noise. The corresponding master equation is given by [18]

$$\frac{\partial P(\boldsymbol{a},t)}{\partial t} = H(\boldsymbol{a})P(\boldsymbol{a},t) \tag{A10}$$

with the master operator $H(\boldsymbol{a})$, whose explicit form is not necessary in the present discussion.

The characteristic function for the time series $\{V_t \equiv V[\boldsymbol{A}(\boldsymbol{t})]\}$, $V[\boldsymbol{A}(t)]$ being a scalar function of the Langevin variable $\boldsymbol{A}(t)$, is given by [19,20]

$$Z_q(t) = \left\langle \exp\left(q\int_{t_0}^{t_0+t} V[\boldsymbol{A}(s)]ds \right) \right\rangle = \int e^{tH_q(\boldsymbol{a})}P_*(\boldsymbol{a})d\boldsymbol{a}, \tag{A11}$$

where the brackets denote the long time average and therefore it is identical to the average over the steady state probability density $P_*(\boldsymbol{a})$. We have defined the generalized master operator H_q by [19,20]

$$H_q(\boldsymbol{a}) = H(\boldsymbol{a}) + qV(\boldsymbol{a}). \tag{A12}$$

Equation (A11) implies that $\phi(q)$ is identical to the largest eigenvalue of $H_q(\boldsymbol{a})$.

3. Chaotic dynamics

The equation of motion is written as

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}). \tag{A13}$$

Let $V_t \equiv V[\boldsymbol{X}(t)]$ is an observed variable, whose characteristic function is given by

$$Z_q(t) = \left\langle \exp\left(q\int_{t_0}^{t_0+t} V[\boldsymbol{X}(s)]ds \right) \right\rangle, \tag{A14}$$

where the brackets represent the long time average. Note that the following relation holds:

$$\exp\left(q\int_{t_0}^{t_0+t} V[\boldsymbol{X}(s)]ds \right) = e^{t\underline{H}_q(\boldsymbol{X})}1, \tag{A15}$$

where $\boldsymbol{X}=\boldsymbol{X}(t_0)$ and $\underline{H}_q$ is the linear operator defined by [21]

$$\underline{H}_q(\boldsymbol{X}) = \underline{H}(\boldsymbol{X}) + qV(\boldsymbol{X}). \tag{A16}$$

The operator $\underline{H}$ is the adjoint operator of the time evolution operator and is defined by

$$\underline{H}(\boldsymbol{X})G = \sum_{\mu} F_{\mu}(\boldsymbol{X})\frac{\partial G}{\partial X_{\mu}}. \tag{A17}$$

Therefore, the characteristic function is written as

$$Z_q(t) = \int e^{t\underline{H}_q(\boldsymbol{X})}1 d\mu(\boldsymbol{X}), \tag{A18}$$

where $\mu(\boldsymbol{X})$ is the invariant measure of the chaotic state. Introducing the invariant density $\rho_*(\boldsymbol{X})$ by $d\mu(\boldsymbol{X})=\rho_*(\boldsymbol{X})d\boldsymbol{X}$, we obtain

$$Z_q(t) = \int e^{tH_q(\boldsymbol{X})}\rho_*(\boldsymbol{X})d\boldsymbol{X}, \tag{A19}$$

where $H[=-\sum_{\mu}(\partial/\partial X_{\mu})F_{\mu}(\boldsymbol{X})]$ and H_q are the adjoint operators respectively of $\underline{H}$ and $\underline{H}_q$. The characteristic function $\phi(q)$ is thus identical to the largest eigenvalue of H_q and equivalently of $\underline{H}_q$.

As shown above, the largest eigenvalue $\lambda_1(q)$ determines the large deviation statistics. It is known that the eigenvalues of H are relevant to double time correlation function of V_t. Previously we proposed a generalized time correlation function to connect the large deviation statistics with the time correlation function [22–24]. The generalized power spectrum $I_q(\omega)$ is expanded as

$$I_q(\omega) = \lim_{t\to\infty}\left\langle \left| \frac{1}{\sqrt{t}}\int_{t_0}^{t_0+t} [V_s - v(q)]e^{-i\omega s}ds \right|^2 \frac{e^{qt\bar{V}_t}}{Z_q(t)} \right\rangle \tag{A20}$$

$$= \sum_{n(\neq 1)} K_n(q)\left[\frac{\gamma_n(q)}{[\gamma_n(q)]^2 + [\omega - \omega_n(q)]^2} + \frac{\gamma_n(q)}{[\gamma_n(q)]^2 + [\omega + \omega_n(q)]^2}\right], \tag{A21}$$

where

$$\gamma_n(q) = \lambda_1(q) - \mathrm{Re}\,\lambda_n(q), \tag{A22}$$

$$\omega_n(q) = \mathrm{Im}\,\lambda_n(q), \tag{A23}$$

and $K_n(q)$'s are expansion constants. The corresponding generalized time correlation function is given by

$$C_q(t) = \sum_{n(\neq 1)} K_n(q)e^{-\gamma_n(q)t}\cos[\omega_n(q)t]. \tag{A24}$$

For $q=0$, $C_0(t)$ is identical to the conventional time correlation function of the time series $\{V_t\}$. By taking the weighted average of the power spectrum of $\{V_t\}$, the generalized power spectrum $I_q(\omega)$ and the generalized time correlation function $C_q(t)$ are capable of picking up different characteristics of time evolution of V_t [22–25].

APPENDIX B: "EQUATIONS OF MOTION" FOR EIGENVALUES AND EIGENSTATES IN DISCRETE-TIME SYSTEMS

Let H be the time evolution operator and V be the observed variable. As will be shown in the subsequent subsections in this appendix, the large deviation statistical quantity is determined by the largest eigenvalue of the generalized operator defined by

$$H_q = He^{qV}. \tag{B1}$$

The eigenvalue equations are expressed as

$$H_q|n\rangle = e^{\lambda_n(q)}|n\rangle, \tag{B2}$$

$$\langle m|H_q = e^{\lambda_m(q)}\langle m|, \tag{B3}$$

where we have used the fact that the eigenvalues in the right and left eigenvalue problems are identical. We thus obtain

$$\langle m|n\rangle = \delta_{m,n}, \tag{B4}$$

$$\langle m|H_q|n\rangle = e^{\lambda_n(q)}\delta_{m,n}, \tag{B5}$$

$$V_{mn}(q) = \langle m|V|n\rangle. \tag{B6}$$

The completeness condition of bras and kets is written as

$$\sum_n |n\rangle\langle n| = 1. \tag{B7}$$

Repeating the derivation in Sec. II, we obtain the equations of motion for $\{\lambda_n(q)\}$, $\{V_{mn}(q)\}$ and $|n\rangle, \langle n|$ as

$$\frac{d\lambda_n(q)}{dq} = V_{nn}, \tag{B8}$$

$$\frac{dV_{mn}(q)}{dq} = \sum_{k(\neq m)} \frac{V_{mk}V_{kn}}{1 - e^{\lambda_k - \lambda_m}} + \sum_{k(\neq n)} \frac{V_{mk}V_{kn}}{e^{\lambda_n - \lambda_k} - 1}, \tag{B9}$$

$$\frac{d|n\rangle}{dq} = \sum_{k(\neq n)} \frac{V_{kn}}{e^{\lambda_n - \lambda_k} - 1}|k\rangle, \tag{B10}$$

$$\frac{d\langle m|}{dq} = \sum_{k(\neq m)} \frac{V_{mk}}{1 - e^{\lambda_k - \lambda_m}}\langle k|. \tag{B11}$$

Since the largest eigenvalue of H is unity, we get the initial condition $\lambda_1(0)=0$.

Consider the case in which the number of eigenstates are finite, being denoted by N. It is easy to show that the quantity defined by

$$h \equiv \sum_n^N V_{nn}(q) \tag{B12}$$

is a conserved quantity. As a result, the following relation holds:

$$\sum_n^N [\lambda_n(q) - \lambda_n(0)] = hq. \tag{B13}$$

The rate of contraction or divergence of the volume of a tiny region in the state space spanned by $\{\lambda_n\}$ and $\{V_{mn}\}$ is given by

$$R(q) = \sum_{m\neq n} \frac{V_{mm} - V_{nn}}{e^{\lambda_n - \lambda_m} - 1} = -\frac{d}{dq}\sum_{m\neq n} \ln|1 - e^{\lambda_m - \lambda_n}|. \tag{B14}$$

1. Finite-state Markovian process

Let us consider the N-state Markovian process given by the evolution equation

$$\boldsymbol{P}(t+1) = \hat{H}\boldsymbol{P}(t) \quad (t = 0,1,2,\ldots) \tag{B15}$$

where $\boldsymbol{P}(t)=[P_1(t),P_2(t),\ldots,P_N(t)]^T$, $P_j(t)$ being the probability that the system is in the jth state at time t. $\hat{H}$ is the transition matrix with the jk element H_{jk} ($\Sigma_{j=1}^N H_{jk}=1$). Let us consider the time series of V_t, which takes the value a_j if the system is in the jth state. The characteristic function for the time series $\{V_t\}$ is given by

$$Z_q(t) = \left\langle \exp\left(q\sum_{s=0}^{t-1} V_s\right)\right\rangle = \sum_{j=1}^N \left(\hat{H}_q^t\boldsymbol{P}_*\right)_j, \tag{B16}$$

where $\boldsymbol{P}_*$ is the steady probability density, and the generalized evolution matrix $\hat{H}_q$ is defined by

$$\hat{H}_q = \hat{H}e^{q\hat{V}}, \tag{B17}$$

where $\hat{V}$ is the matrix with the jk element $V_{jk}=a_j\delta_{j,k}$. For large t, we get $Z_q(t) \sim e^{\phi(q)t}$. Thus, we find that the large deviation statistical characteristic function is identical to the logarithm of the largest eigenvalue of $\hat{H}_q$.

The eigenvalue equations of $\hat{H}_q$ are written as

$$(\hat{H}_q \boldsymbol{e}_n)_j = e^{\lambda_n(q)}(\boldsymbol{e}_n)_j, \quad (B18)$$

$$(\underline{\boldsymbol{e}}_m \hat{H}_q)_j = (\underline{\hat{H}}_q \underline{\boldsymbol{e}}_m)_j = e^{\lambda_m(q)}(\underline{\boldsymbol{e}}_m)_j, \quad (B19)$$

where $\underline{\hat{H}}_q$ is the adjoint matrix of $\hat{H}_q$ ($\underline{\hat{H}}_q = \hat{H}_q^T$). The element V_{mn} is given by

$$V_{mn}(q) = \underline{\boldsymbol{e}}_m \hat{V} \boldsymbol{e}_n = \sum_{j=1}^{N} a_j (\underline{\boldsymbol{e}}_m)_j (\boldsymbol{e}_n)_j. \quad (B20)$$

The completeness condition is written as

$$\sum_n (\underline{\boldsymbol{e}}_n)_k (\boldsymbol{e}_n)_j = \delta_{j,k}. \quad (B21)$$

The eigenvalues $\{\lambda_n\}$ and the elements $\{V_{mn}\}$ obey the equations of motion (B8) and (B9).

2. Chaotic maps

Let us consider the chaotic map

$$\boldsymbol{X}_{t+1} = \boldsymbol{F}(\boldsymbol{X}_t) \quad (B22)$$

($t=0,1,2,3,\ldots$). The probability density $\rho_t(\boldsymbol{x})$ of $\boldsymbol{X}_t$ obeys

$$\rho_{t+1}(\boldsymbol{x}) = \int d\boldsymbol{y} H_{\boldsymbol{x},\boldsymbol{y}} \rho_t(\boldsymbol{y}) \equiv H\rho_t(\boldsymbol{x}), \quad (B23)$$

$$H_{\boldsymbol{x},\boldsymbol{y}} = \delta[\boldsymbol{x} - \boldsymbol{F}(\boldsymbol{y})]. \quad (B24)$$

The adjoint operator $\underline{H}$ of the Frobenius-Perron operator H has the element

$$\underline{H}_{\boldsymbol{x},\boldsymbol{y}} = \delta[\boldsymbol{y} - \boldsymbol{F}(\boldsymbol{x})], \quad (B25)$$

$$\underline{H}G(\boldsymbol{x}) = \int d\boldsymbol{y} \underline{H}_{\boldsymbol{x},\boldsymbol{y}} G(\boldsymbol{y}) = G[\boldsymbol{F}(\boldsymbol{x})]. \quad (B26)$$

For the observed time series of a scalar variable $V_t \equiv V(\boldsymbol{X}_t)$, the characteristic function is given by

$$Z_q(t) = \left\langle \exp\left(q \sum_{s=0}^{t-1} V_s \right) \right\rangle = \int [\underline{H}_q(\boldsymbol{x})]^t 1 d\mu(\boldsymbol{x}) = \int [H_q(\boldsymbol{x})]^t \rho_*(\boldsymbol{x}) d\boldsymbol{x} \quad (B27)$$

with $\mu(\boldsymbol{x})$ and $\rho_*(\boldsymbol{x})$ being, respectively, the invariant measure and the invariant density. Here, the generalized Frobenius-Perron operator H_q and its adjoint operator $\underline{H}_q$ have been defined by

$$H_q G(\boldsymbol{x}) = \int d\boldsymbol{y} (H_q)_{\boldsymbol{x},\boldsymbol{y}} G(\boldsymbol{y}), \quad (B28)$$

$$H_q(\boldsymbol{x}) = H(\boldsymbol{x}) e^{qV(\boldsymbol{x})}, \quad (B29)$$

$$(H_q)_{\boldsymbol{x},\boldsymbol{y}} = \delta[\boldsymbol{x} - \boldsymbol{F}(\boldsymbol{y})] e^{qV(\boldsymbol{y})} \quad (B30)$$

and

$$\underline{H}_q G(\boldsymbol{x}) = \int d\boldsymbol{y} (\underline{H}_q)_{\boldsymbol{x},\boldsymbol{y}} G(\boldsymbol{y}), \quad (B31)$$

$$\underline{H}_q(\boldsymbol{x}) = e^{qV(\boldsymbol{x})} \underline{H}(\boldsymbol{x}), \quad (B32)$$

$$(\underline{H}_q)_{\boldsymbol{x},\boldsymbol{y}} = (H_q)_{\boldsymbol{y},\boldsymbol{x}} = e^{qV(\boldsymbol{x})} \underline{H}_{\boldsymbol{x},\boldsymbol{y}}. \quad (B33)$$

For large t, $Z_q(t)$ obeys $Z_q(t) \sim e^{\phi(q)t}$. Therefore, the large deviation characteristic function $\phi(q)$ is identical to the logarithm of the largest eigenvalue of $\underline{H}_q$ and H_q.

The eigenvalue equations for H_q and $\underline{H}_q$ are written as

$$(H_q e_n)(\boldsymbol{x}) = \int d\boldsymbol{y} (H_q)_{\boldsymbol{x},\boldsymbol{y}} e_n(\boldsymbol{y}) = e^{\lambda_n} e_n(\boldsymbol{x}), \quad (B34)$$

$$(\underline{e}_m H_q)(\boldsymbol{x}) = \int d\boldsymbol{y} \underline{e}_m(\boldsymbol{y}) (H_q)_{\boldsymbol{y},\boldsymbol{x}} = (\underline{H}_q \underline{e}_m)(\boldsymbol{x}) = e^{\lambda_m} \underline{e}_m(\boldsymbol{x}). \quad (B35)$$

The orthogonality of eigenstates and the elements are, respectively, given by

$$\delta_{m,n} = \int d\boldsymbol{x} \underline{e}_m(\boldsymbol{x}) e_n(\boldsymbol{x}), \quad (B36)$$

$$V_{mn}(q) = \int d\boldsymbol{x} \underline{e}_m(\boldsymbol{x}) V(\boldsymbol{x}) e_n(\boldsymbol{x}). \quad (B37)$$

The logarithmic eigenvalues $\{\lambda_n\}$ and the elements $\{V_{mn}\}$, respectively, obey the equations of motion (B8) and (B9). Furthermore, the completeness condition of eigenstates is written as

$$\sum_n \underline{e}_n(\boldsymbol{y}) e_n(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{y}). \quad (B38)$$

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